

Multipartite quantum correlations: symplectic and algebraic geometry approach

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Abstract

We review a geometric approach to classification and examination of quantum correlations in composite systems. Since quantum information tasks are usually achieved by manipulating spin and alike systems or, in general, systems with a finite number of energy levels, classification problems are usually treated in frames of linear algebra. We proposed to shift the attention to a geometric description. Treating consistently quantum states as points of a projective space rather than as vectors in a Hilbert space we were able to apply powerful methods of differential, symplectic and algebraic geometry to attack the problem of equivalence of states with respect to the strength of correlations, or, in other words, to classify them from this point of view. Such classifications are interpreted as identification of states with ‘the same correlations properties’ i.e. ones that can be used for the same information purposes, or, from yet another point of view, states that can be mutually transformed one to another by specific, experimentally accessible operations. It is clear that the latter characterization answers the fundamental question ‘what can be transformed into what *via* available means?’. Exactly such an interpretation, i.e. in terms of mutual transformability can be clearly formulated in terms of actions of specific groups on the space of states and is the starting point for the proposed methods.

1 Introduction

Quantum entanglement - a direct consequence of linearity of quantum mechanics and the superposition principle - is one of the most intriguing phenomena distinguishing quantum and classical description of physical systems. Quantum correlated (e.g., entangled) states of composite systems possess features unknown in the classical world, like the seemingly paradoxical nonlocal properties exhibited by the famous Einstein-Podolsky-Rosen analysis of completeness of the quantum theory. Recently, with the development of quantum information theory they came to prominence as the main resource for several applications aiming at speeding up and making more secure information transfers (see, e.g., [1]). A novel kind of quantum correlations, called *quantum discord*, different from entanglement, but also absent in the macroscopic world, was discovered [2], [3] adding one more element to “mysteries of quantum mechanics” as seen from the classical point of view.

Although typically a quantum system, as, e.g., a harmonic oscillator or a hydrogen atom, is described in terms of an infinitely dimensional Hilbert space, for most quantum-information applications the restriction to finite dimensions suffices, since usually the active role in information processing play only spin degrees of freedom or only few energy levels are excited during the evolution.

From the mathematical point of view such finite-dimensional quantum mechanics seems to mount a smaller challenge than in the infinite-dimensional case - the tool of choice is here linear algebra rather than functional analysis. Nevertheless, understanding of correlations in multipartite finite dimensional quantum systems is still incomplete, especially for systems with many components or consisting of particles governed by quantum statistics (bosons, fermions) [4].

The statistical interpretation of quantum mechanics disturbs a bit the simple linear-algebraic approach to quantum mechanics - vectors of state (elements of a finite-dimensional Hilbert space \mathcal{H}) should be of the unit norm. Obviously, physicist are accustomed to cope with this problem in a natural way by “normalizing the vector and neglecting the global phase”. Nevertheless, it is often expedient to implement this prescription by adopting a suitable mathematical structure, the projective space $\mathbb{P}(\mathcal{H})$, already from the start. The projective space is obtained from the original Hilbert space \mathcal{H} by identifying vectors¹ differing by a scalar, complex, non-zero factor, $|\psi\rangle \cong c|\psi\rangle$. We will denote elements (points) of $\mathbb{P}(\mathcal{H})$ by u, v, x , etc. and, if we want to identify a particular equivalence class of a vector $|\psi\rangle$, by $[\psi]$, etc.

Obviously, both approaches, the linear-algebraic (plus normalization and neglecting the global case) picture and the projective one are equivalent. Following the former we loose linearity, so which advantages we could expect instead? We answered this question in our paper [5], where we propose, by working in the projective space, to apply completely new (in this context) techniques to analyze the phenomenon of entanglement. The approach has given a deeper insight into the unexpectedly rich geometric structure of the space of states and enabled the use of recently developed advanced methods of complex differential, algebraic and symplectic geometry.

The most efficient characterization of quantum correlations is achieved by identifying states that are ‘equally correlated’ or, in other words, states that can be mutually transformed *via* methods allowed by rules of quantum mechanics without destroying quantum correlations. This is the case of pure quantum operations that are *local*, i.e. restricted to act independently on each subsystem. As required by quantum mechanics, such operations are unitary in the relevant Hilbert space \mathcal{H} , but form only a subgroup of the whole group $U(\mathcal{H})$ as limited by the restriction to subsystems. For these reasons they are called *local unitary* (LU) operations. A complementary insight is provided by adding more operations, like measurements performed locally, i.e., each restricted to a single subsystem, and possible communication among subsystems *via* classical means of communication. Such operations are dubbed LOCC - *Local Operations with Classical Communication*. They usually destroy purely quantum correlations, but in any case do not increase them, so if one state can be transformed to another one *via* LOCC transformation, the latter is not more (and usually less) correlated than the former. This allows quantifying correlations by constructing *entanglement measures*, quantities characterizing states that remain unchanged under LU operations and do not increase under LOCC ones. Technically it is convenient to group together states related to each other by invertible stochastic local

¹We will use exchangeably the Dirac notation, $|\psi\rangle$, etc., and the short one ψ etc. for elements (vectors) of \mathcal{H} .

operations and classical communication (SLOCC) Two states, $|\phi\rangle$ and $|\psi\rangle$ belong to the same SLOCC class if both transformations, $|\psi\rangle \rightarrow |\phi\rangle$ and $|\phi\rangle \rightarrow |\psi\rangle$ can be achieved by means of LOCC with non-zero probability. Hence, in general, if two states belong to two distinct SLOCC classes, they have different correlation properties and might be not exchangeable for achieving the same quantum informational tasks.

As mentioned above the LU operations form a subgroup K of the whole unitary group $U(\mathcal{H})$ of the relevant Hilbert space \mathcal{H} (since the global phase of a state is irrelevant, we can use the special unitary group $SU(\mathcal{H})$ instead). The SLOCC also form a group G that happens to be the complexification of K , i.e., $G = K^{\mathbb{C}}$. In contrast to K , which is compact as a subgroup of the unitary group, G is not compact. The exact forms of K and, consequently, G depend on the problem in question (number and dimensionality of subsystems, distinguishability of particles constituting them). The actions of K and G on \mathcal{H} are naturally and easily transferred to the projective space $\mathbb{P}(\mathcal{H})$ via $U[\psi] = [U\psi]$.

The projective space has a rich geometrical structure. In particular, in a natural way, it is a Kähler manifold, i.e., a complex symplectic Riemannian manifold on which all three structures are compatible. Symplecticity means that it can be treated as a kind of a classical phase space for a Hamiltonian system. The action of a group on such a manifold reflects symmetries of the dynamical system, and the methods of analysis of such systems can be borrowed from classical mechanics. In modern formulation of classical mechanics (see, e.g., [6], [7]) one employs the full power of symplectic geometry.

So far the above outlined geometric treatment of correlations was restricted to pure states. There exists, however, its natural extension to mixed states. Instead of the projective space $\mathbb{P}(\mathcal{H})$ one considers another manifold - an orbit of the whole $SU(\mathcal{H})$ group through a chosen mixed state (i.e., the set of all mixed density matrices obtained for the chosen one ρ via by unitary conjugations, $U\rho U^\dagger$, $U \in SU(\mathcal{H})$), in other words all density matrices with the same spectrum. Such an orbit possesses a natural Kählerian structure given by the so called Kirillov-Kostant-Souriau form. This observation allows to apply the same geometric techniques as in the pure-state case.

2 Symplectic structures in spaces of states

As outlined in previous sections, one of the basic problems in the theory of quantum correlations is the classification of states with respect to local operations performed independently on subsystems of a given system by two classes of such operations LU and SLOCC, both being groups acting on the space of states, a unitary one K and its complexification $G = K^{\mathbb{C}}$, respectively. The space of pure states is the projective space $\mathbb{P}(\mathcal{H})$, and for mixed states, the space of spectral (i.e., possessing the same eigenvalues) density matrices. The underlying Hilbert space for a composite system has a form the tensor product of the Hilbert spaces of the subsystems or, in the case of indistinguishable components, an appropriate (anti)-symmetrization of the product.

In both cases of pure and mixed states the above described spaces of states have a natural geometric structure of a symplectic manifolds (see Appendix for a brief description of relevant mathematical concepts used in the paper).

For applications in the theory of quantum correlations we will consider the following particular instances.

1. $M = \mathbb{P}(\mathcal{H})$ (the complex projective space), and

- (a) for pure states of L distinguishable particles the Hilbert space is the tensor product, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_L$, where \mathcal{H}_i is the N_i -dimensional Hilbert space of i -th particle, the action of $K = SU(N_1) \times \dots \times SU(N_L)$ on \mathcal{H} is defined in the natural way in terms of the tensor product, i.e. $U_1 \otimes \dots \otimes U_N \cdot (\psi_1 \otimes \dots \otimes \psi_n) = U_1 \psi_1 \otimes \dots \otimes U_N \psi_n$, the complexified group is thus $G = K^{\mathbb{C}} = SL(N_1, \mathbb{C}) \times \dots \times SL(N_L, \mathbb{C})$, and the Lie algebras read $\mathfrak{k} = \mathfrak{su}(N_1) \oplus \dots \oplus \mathfrak{su}(N_L)$, $\mathfrak{g} = \mathfrak{sl}(N_1, \mathbb{C}) \oplus \dots \oplus \mathfrak{sl}(N_L, \mathbb{C})$;
- (b) for pure states of L d -state bosons $\mathcal{H} = S^L \mathcal{H}_1$ (the L -th symmetrized tensor power of \mathcal{H}_1), where \mathcal{H}_1 is the Hilbert space of a single boson, the group $K = SU(d)$ acts diagonally, $U \cdot (\psi_1 \vee \dots \vee \psi_L) = U \psi_1 \vee \dots \vee U \psi_L$, and, consequently, $G = K^{\mathbb{C}} = SL(d, \mathbb{C})$, $\mathfrak{k} = \mathfrak{su}(d)$, $\mathfrak{g} = \mathfrak{sl}(d, \mathbb{C})$;
- (c) for pure states of L d -state fermions, $\mathcal{H} = \bigwedge^L \mathcal{H}_1$ (the L -th anti-symmetrized tensor power of \mathcal{H}_1), where \mathcal{H}_1 is the Hilbert space of a single fermion, the group $K = SU(d)$ again acts diagonally (i.e. by the same element of the group on each factor in the anti-symmetrized product), $G = K^{\mathbb{C}} = SL(d, \mathbb{C})$, algebras $\mathfrak{k} = \mathfrak{su}(d)$, $\mathfrak{g} = \mathfrak{sl}(d, \mathbb{C})$.

2. Isospectral density matrices \mathcal{O}_ρ for systems of distinguishable particles with the Hilbert space and the groups K and G as in 1(a) above but now acting *via* conjugations (the adjoint action) on the Lie algebra \mathfrak{k} . In fact density matrices are not exactly elements of respective Lie algebras since the latter are traceless. Nevertheless a shift by a constant multiple of the identity yields the desired normalization to $\text{Tr} \rho = 1$, and the requirement of positivity restricts them to a subset of so ‘shifted’ Lie algebra (see below).

For $\mathbb{P}(\mathcal{H})$ the Kirillov-Kostant-Souriau form providing a symplectic structure, calculated at a point $[v] \in \mathbb{P}(\mathcal{H})$ reads

$$\omega(\hat{\xi}_1, \hat{\xi}_2) = \frac{-i \langle v | [\xi_1, \xi_2] v \rangle}{2 \langle v | v \rangle}, \quad (1)$$

whereas for \mathcal{O}_ρ at a density matrix σ ,

$$\omega(\hat{\xi}_1, \hat{\xi}_2) = -\frac{i}{2} \text{Tr}(\sigma [\xi_1, \xi_2]), \quad (2)$$

The momentum map $\mu : \mathbb{P}(\mathcal{H}) \rightarrow \mathfrak{k}$ for pure states of L distinguishable particles is given by

$$\mu([v]) = \frac{i}{2} \left\{ \rho_1([v]) - \frac{1}{N_1} I_{N_1}, \rho_2([v]) - \frac{1}{N_2} I_{N_2}, \dots, \rho_L([v]) - \frac{1}{N_L} I_{N_L} \right\}, \quad (3)$$

where $\rho_i([v])$ is the i -th reduced one-particle density matrix of a state $[v] \in \mathbb{P}(\mathcal{H})$ and I_{N_i} is the identity operator on the N_i -dimensional Hilbert space \mathcal{H}_i . For pure states of L d -state indistinguishable particles (bosons or fermions) it reads $\mu([v]) = \frac{i}{2} (\rho_1([v]) - \frac{1}{d} I)$, $\rho_1([v])$, and for the case of mixed states on a single orbit \mathcal{O}_ρ by the formula similar to (3) but with ρ_i being now the reduced one-particle density matrices of a state $\sigma \in \mathcal{O}_\rho$.

3 Local unitary equivalence of quantum states

In all above considered cases the momentum map relates a state to its one-particle reduced density matrices. This observation paves a way of relating the local unitary equivalence to the properties of the reduced states or, in other words, to properties of the momentum map.

Since K is a compact group its orbits are closed. To check if two states belong to the same orbit it suffices to find the (finitely generated [8]) ring of K -invariant polynomials, i.e. polynomials that are constant on orbits of K , since each orbit is uniquely determined by the values of all generating polynomials in an arbitrary point of it. The equivariant momentum map, $\mu : M \rightarrow \mathfrak{k}$, maps K -orbits in the space of states M onto orbits of the adjoint action of K in \mathfrak{k} . Consequently, each Ad_K -invariant polynomial $p : \mathfrak{k} \rightarrow \mathbb{R}$ on \mathfrak{k} , when composed with μ , gives a K -invariant polynomial $P = p \circ \mu : M \rightarrow \mathbb{R}$ on M . The invariant polynomials for the adjoint action of $SU(N)$ are generated by traces of powers not larger than N of $X \in \mathfrak{su}(N)$. Combining this with the fact that the momentum map is given by reduced one-particle density matrices we conclude that traces of their powers are K -invariant polynomials on M . If the pre-image of every adjoint orbit from $\mu(M) \subset \mathfrak{k}$ consists of exactly one K -orbit in M , a K -orbit on M (i.e., a set of equally correlated states) can be identified by the traces of powers of the reduced one-particle density matrices. Since the traces of powers of a matrix determine its spectrum (and *vice versa*) we can decide upon local unitary equivalence of states by examining the spectra of their reduced one-particle density matrices, hence, in principle by measurements restricted performed independently in each laboratory only.

Alas, the above described situation is quite exceptional. Typically many K -orbits are mapped onto one adjoint orbit. Hence, even if one-particle density matrices of two states have the same spectra we need additional K -invariant polynomials to decide their K -equivalence [9], [10]. The number of the additional polynomials characterizes, in a certain sense, the amount of additional information that can not be obtained from local measurements, but needs to be inferred from non-local ones, i.e., involving effectively more than one subsystem. The whole problem can be looked upon as a quantum version of the classical ‘marginal problem’, where we try to recover a probability distribution from its marginals (here, recover a quantum state from its reduced density matrices) [11].

Let us, however, concentrate for a moment on cases (exceptional, as it follows from the above remarks), when we infer local unitary equivalence upon examining the spectra of the reduced states.

3.1 Spherical embeddings and determination of the local equivalence from the spectra of the reduced states

As pointed above, the situation when the set of K -invariant polynomials on M is given by the composition of Ad_K -invariant polynomials on \mathfrak{k} with the momentum map $\mu : M \rightarrow \mathfrak{k}$ occurs only when the pre-image of every adjoint orbit from $\mu(M) \subset \mathfrak{k}$ is exactly one K -orbit in M . Let us thus consider the fibre of the momentum map μ over $\mu(x) \in \mathfrak{k}$,

$$\mathcal{F}_x := \{z \in M : \mu(z) = \mu(x)\}, \quad (4)$$

i.e. the set of all points in M that are mapped to the point $\mu(x)$ by the momentum map. The position of \mathcal{F}_x with respect to the orbit $K.x$ is of the key importance for the above situation. Orbits of K in M are in 1-1 correspondence with the adjoint orbits in $\mu(M)$ if

and only if the fibres \mathcal{F}_x are contained in orbits $K.x$, since in this case all points mapped to $\mu(x)$ are on the same orbit (that can be thus identified, when we know $\mu(x)$).

It is easy to see that the tangent space $T_y\mathcal{F}_x$ at y is ω -orthogonal to the tangent space $T_y\mathcal{O}_x$ at the same point y , i.e., for each $a \in T_y\mathcal{F}_x$ and $b \in T_y\mathcal{O}_x$ we have $\omega(a, b) = 0$. Indeed, let $\{\xi_k\}$, $k = 1, \dots, d = \dim \mathfrak{g}$, be a basis in the Lie algebra \mathfrak{g} . The corresponding vector fields $\hat{\xi}_k$ at x (c.f. Eq. (18) in the Appendix) span the tangent space $T_x\mathcal{O}_x$ to the orbit through x at x . On the other hand, the fiber \mathcal{F}_x is a common level set of the functions μ_{ξ_k} ,

$$\mathcal{F}_x = \{z \in M : \mu_{\xi_k}(z) = c_k\}, \quad c_k = \mu_{\xi_k}(x), \quad k = 1, \dots, d, \quad (5)$$

which means that the derivatives of μ_{ξ_k} vanish on the tangent to \mathcal{F}_x , $d\mu_{\xi_k}(a) = 0$ for $a \in T_x\mathcal{F}_x$.

Define now the kernel of $d\mu$,

$$\text{Ker}_x(d\mu) := \{a \in T_xM : d\mu_{\xi_k}(x)(a) = 0, k = 1, \dots, d\}. \quad (6)$$

From the definition of the momentum map (see Appendix) we have

$$d\mu_{\xi_k}(a) = \omega(\hat{\xi}_k, a). \quad (7)$$

Hence $a \in \text{Ker}_x(d\mu)$ if and only if a is ω -orthogonal to all $\hat{\xi}_k$. Since $\hat{\xi}_k$ span $T_x\mathcal{O}_x$ we have

$$\text{Ker}_x(d\mu) = (T_x\mathcal{O}_x)^{\perp\omega}, \quad (8)$$

where by $X^{\perp\omega}$ we denote the space of ω -orthogonal vectors to X . Combining this with the observation above that $d\mu_{\xi_k}$ vanish on $T_x\mathcal{F}_x$, we find that $T_x\mathcal{F}_x \subset \text{Ker}_x(d\mu)$ and, consequently, $T_x\mathcal{F}_x \subset (T_x\mathcal{O}_x)^{\perp\omega}$. The above reasoning clearly does not depend on the choice of a particular point in \mathcal{F}_x , i.e., as announced,

$$T_y\mathcal{F}_x \subset (T_y\mathcal{O}_x)^{\perp\omega}, \quad y \in \mathcal{F}_x. \quad (9)$$

A submanifold P of a symplectic manifold M is called coisotropic if for arbitrary $y \in P$ we have $(T_yP)^{\perp\omega} \subset T_yP$. We conclude thus that if \mathcal{O}_x is coisotropic then $\mathcal{F}_x \subset \mathcal{O}_x$. Indeed from (9) and the coisotropy of \mathcal{O}_x at each $y \in \mathcal{F}_x$ we have $T_y\mathcal{F}_x \subset T_y\mathcal{O}_x$. Hence, in this case examining whether some y belongs to \mathcal{O}_x (and, consequently whether y and x are LU-equivalent) reduces to checking whether their corresponding one-particle reduced states have the same spectra [12].

In order to characterize all systems for which LU-equivalence can be decided using the spectra of one-particle density matrices we need to identify those whose (at least) generic K -orbit is coisotropic. As it turns out, such systems need to satisfy some group theoretic conditions [14]. To reach the final solution we have to study not only the action of K but also of its complexification $G = K^{\mathbb{C}}$ on M . Note that the group G is much bigger than K and therefore the number of G -orbits is smaller than number of K -orbits in M . If G has an open dense orbit on M then we call M *almost homogenous manifold* [13]. Such M has a finite number of G -orbits. The almost homogeneity of M with respect to G -action is a necessary condition for deciding K -equivalence on M using the momentum map [14]. It is, however, not a sufficient condition, what can be seen from the example of the three-qubit system, where there are exactly 6 orbits of $G = SL(2, \mathbb{C})^{\times 3}$ action, but states $x_1 = \sqrt{\frac{2}{3}}|000\rangle + \frac{1}{\sqrt{3}}|111\rangle$ and $x_2 = \frac{1}{\sqrt{3}}(|000\rangle + |010\rangle + |001\rangle)$, where $\{|0\rangle, |1\rangle\} \subset \mathbb{C}^2$

is an orthonormal basis in \mathbb{C}^2 , satisfy $\mu(x_1) = \mu(x_2)$ but are not K -equivalent as they belong to different G -orbits [15].

An important role in the formulation of the sufficient condition is played by the Borel subgroup of the group G . By definition a Borel subgroup B is a maximal connected solvable subgroup of the group G . For example, for $G = \text{SL}_N(\mathbb{C})$, the group of upper-triangular matrices, that is a stabilizer of the standard full flag in \mathbb{C}^N , (i.e., a collection of subspaces with the dimension growing by one, such that the given one includes all subspaces with smaller dimensions),

$$0 \subset \text{Span}\{e_1\} \subset \text{Span}\{e_1, e_2\} \subset \dots \subset \text{Span}\{e_1, \dots, e_{N-1}\} \subset \text{Span}\{e_1, \dots, e_N\} = \mathcal{H}$$

where e_1, \dots, e_N is a basis in \mathcal{H} , is an example of a Borel subgroup. Generally, any two Borel subgroups are conjugated by an element of G . Therefore, in the considered example, B is a Borel subgroup of G if and only if when it stabilizes some standard full flag.

The crucial notion for the K -equivalence problem is the notion of a *spherical space*. If G is a group and H its subgroup we call $\Omega = G/H$, i.e., the space obtained from G by identifying points connected by elements of H , a *G -homogenous space*. In this space the group G acts in a natural way, simply by the group product followed by identification of elements connected by elements of H , hence we can consider actions of G or its subgroups on Ω , in particular of a Borel subgroup. This leads to the following definition. A G -homogenous space $\Omega = G/H$ is a *spherical homogenous space* if and only if some (and therefore every) Borel subgroup $B \subset G$ has an open dense orbit in Ω .

Now observe that an orbit of G in the initially considered manifold M through a point x can be identified with some G -homogenous space. Indeed we can take as H the subgroup that stabilizes x , than all other points on the orbit are obtained by actions of G that effectively move x .

All these considerations allow the following definition. If G has an open dense orbit $\Omega = G/H$ in M and Ω is a spherical homogenous space then M is called a *spherical embedding* of $\Omega = G/H$. Such M is also called *almost homogenous spherical space* (with respect to the action of G). Its relevance stems from the following Brion's theorem [16],

Theorem 1 (Brion) *Let K be a connected compact Lie group acting on connected compact Kähler manifold (M, ω) by a Hamiltonian action and let $G = K^{\mathbb{C}}$. The following are equivalent*

1. M is a spherical embedding of the open G -orbit.
2. For every $x \in M$ the fiber \mathcal{F}_x is contained in $K.x$.

In other words, the second point above assures that the momentum map separates all K -orbits, i.e., from $\mu(x) = \mu(y)$ it follows $y \in K.x$. In our case the group K is the local unitary group, i.e., the group of the product of the local unitaries. The conclusion is that if the relevant (\mathcal{H}) (the space of states of the whole system) is a spherical embedding of an open orbit of the corresponding group of the product of the local special linear groups (i.e. the complexification of the local unitary group) then LU-equivalence can be decided upon spectra of the reduced one-particle states.

As it should be clear from the above presented reasoning we thus need to find open dense orbit of G in $\mathbb{P}(\mathcal{H})$ and show that $\mathbb{P}(\mathcal{H})$ is a spherical embedding of it (in short, that the action of G on $\mathbb{P}(\mathcal{H})$ is *spherical*). In [14] we use Brion's theorem and show that open dense orbits of the Borel subgroup exist for systems of two fermions, two bosons and two distinguishable particles in arbitrary dimensions. Therefore, for such system the LU-equivalence can be decided using reduced one-particle density matrices.

3.2 Exceptional states

From the previous section we conclude that typically we are confronted with undecidability of local unitary equivalence upon examining the spectra of reduced one-particle density matrices. The problem can be looked upon from another point of view, using tools borrowed from algebraic geometry, by exhibiting causes for such a situation. In particular, one can show that an obstacle is the existence of so-called exceptional states. To describe them we define the rank of a state [17]. To this end we consider the images of three maps corresponding to the cases of distinguishable particles, bosons and fermions, respectively,

1. The Segre map, $\text{Seg}_L : \mathbb{P}(\mathcal{H}_1) \times \dots \times \mathbb{P}(\mathcal{H}_L) \longrightarrow \mathbb{P}(\mathcal{H})$, $\text{Seg}_L([v_1], \dots, [v_L]) \longmapsto [v_1 \otimes \dots \otimes v_L]$.
2. The Veronese map, $\text{Ver}_L : \mathbb{P}(\mathcal{H}_1) \longrightarrow \mathbb{P}(\mathcal{H})$, $\text{Ver}_L[v] \longmapsto [v^{\otimes L}]$.
3. The Plücker map, $\text{Pl}_L : \text{Gr}(L, \mathcal{H}_1) \longrightarrow \mathbb{P}(\mathcal{H})$, where $\text{Gr}(L, \mathcal{H}_1)$ is the Grassmanian, i.e., the space of all L -dimensional subspaces of \mathcal{H}_1 , $\text{Pl}_L(\text{span}\{u_1, \dots, u_L\}) \longmapsto [u_1 \wedge \dots \wedge u_L]$.

The rank of a state with respect to \mathbb{X} is defined as

$$\text{rk}[\psi] = \text{rk}_{\mathbb{X}}[\psi] = \min\{r \in \mathbb{N} : \psi = x_1 + \dots + x_r, [x_j] \in \mathbb{X}\}, \quad (10)$$

where \mathbb{X} is the image of the corresponding map (1.-3. above). In words, it means that the rank is the minimal number of states needed to obtain the state in question from states in \mathbb{X} .

The sets of states of rank r will be denoted by $\mathbb{X}_r = \{[\psi] \in \mathbb{P}(\mathcal{H}) : \text{rk}[\psi] = r\}$. They are not closed² and it turns out that there are states in $\mathbb{P}(\mathcal{H})$ of a certain rank r that can be approximated with an arbitrary precision by states of a lower rank.

An example of an exceptional state is a 3-qubit W -state of rank 3, $|W\rangle = \frac{1}{\sqrt{3}}(|011\rangle + |101\rangle + |110\rangle)$. It can be obtained as the limit of a sequence of rank-2 states obtained from the GHZ state, $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. Indeed in $\mathbb{P}(\mathcal{H})$ we have,

$$A(a)^{\otimes 3}[GHZ] \xrightarrow{a \rightarrow 0} [W], \quad A(a) = \frac{1}{\sqrt{2}} \begin{pmatrix} a & a \\ -a^{-1} & a^{-1} \end{pmatrix} \quad (11)$$

It was proved [18] that their existence is an obstacle preventing from inferring the local unitary equivalence of states from the image of the momentum map, i.e., from the spectra of the reduced one-particle density matrices.

The main theorem of [18] reads:

Theorem 2 *Suppose that we have one of the following three configurations of a state space \mathcal{H} , a complex reductive Lie group G acting irreducibly on \mathcal{H} , and a variety of coherent states $\mathbb{X} \subset \mathbb{P}(\mathcal{H})$, which is the unique closed G -orbit in the projective space $\mathbb{P}(\mathcal{H})$.*

- (i) $\mathcal{H}_D = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_L$, $G_D = GL(\mathcal{H}_1) \times \dots \times GL(\mathcal{H}_L)$, $\mathbb{X} = \text{Segre}(\mathbb{P}(\mathcal{H}_1) \times \dots \times \mathbb{P}(\mathcal{H}_L))$.
- (ii) $\mathcal{H}_B = S^L(\mathcal{H}_1)$, $G = GL(\mathcal{H}_1)$, $\mathbb{X} = \text{Ver}_L(\mathbb{P}(\mathcal{H}_1))$.
- (iii) $\mathcal{H}_F = \bigwedge^L \mathcal{H}_1$, $G = GL(\mathcal{H}_1)$, $\mathbb{X} = \text{Pl}(\text{Gr}(L, \mathcal{H}_1))$.

Then the action of G on $\mathbb{P}(\mathcal{H}_{B,F})$ (resp. G_D on $\mathbb{P}(\mathcal{H}_D)$) is spherical if and only if there are no exceptional states in $\mathbb{P}(\mathcal{H}_{B,F})$ (resp. $\mathbb{P}(\mathcal{H}_D)$) with respect to \mathbb{X} . In other words, sphericity of the representation is equivalent to the property that states of a given rank cannot be approximated by states of lower rank.

²in the Zariski topology

Combining this theorem with the results described in the previous section we infer that the existence of exceptional states is an obstacle for deciding LU-equivalence using the momentum map i.e., upon reduced spectra.

3.3 Kirwan polytopes

As already mentioned above, information about the spectra of the reduced one-particle density matrices does not allow to decide whether two states of the whole system belong to the same orbit of the local unitary transformation, i.e. whether they have the same correlation properties. The condition $\mu(K.[\psi]) = \mu(K.[\phi])$, where $K.[\phi]$ denotes the orbit in $\mathbb{P}(\mathcal{H})$ of K through $[\psi]$, etc., is still a necessary one for the K -equivalence of states $[\phi]$ and $[\psi]$. The image of the momentum map, $\mu(M)$ consists of adjoint orbits in \mathfrak{k} . Each adjoint orbit intersects the Cartan subalgebra \mathfrak{t} (maximal commutative subalgebra of \mathfrak{k} at a finite number of points. This statement expresses the fact that each matrix from \mathfrak{k} can be diagonalized by the adjoint action (conjugation) of an element of the group K (just like a (anti-)Hermitian matrix can be diagonalized by some unitary transformation). The Cartan subalgebra, in this case, is represented by diagonal matrices, but the eigenvalues (diagonal elements at the point of intersection) are ordered in some specific way. By permutations we can order them differently, reaching in this way some other point of intersection. Thus different intersection points are connected by the action of some subgroup of the permutation group - the Weyl group. To make the situation unambiguous we impose a concrete order, for example nonincreasing, which means that we choose the intersection point that belongs to a subset of diagonal matrices (i.e., elements of \mathfrak{t} with nonincreasing diagonal entries), called the positive Weyl chamber and denoted by \mathfrak{t}_+ . Now we can define $\Psi : M \rightarrow \mathfrak{t}_+$ to be the map satisfying $\Psi([\phi]) = \mu(K.[\phi]) \cap \mathfrak{t}_+$. It assigns to a state $[\phi]$ the ordered spectra of the (shifted) one-particle reduced density matrices. By taking the intersection of the whole image of the momentum map, $\mu(M)$, with the positive Weyl chamber, \mathfrak{t}_+ , one obtains the set $\Psi(M) = \mu(M) \cap \mathfrak{t}_+$ parametrizing the orbits by elements of \mathfrak{t}_+ [7]. The convexity theorem of the momentum map [19, 20, 21] states that $\Psi(M)$ is a convex polytope, referred to as the Kirwan polytope.

The necessary condition for states $[\phi_1]$ and $[\phi_2]$ to be K -equivalent, can be therefore formulated as $\Psi([\phi_1]) = \Psi([\phi_2])$. To decide whether they are really equivalent we need to inspect additional invariants. eg., additional invariant polynomials. It is shown in [10] that for L -qubit states satisfying the necessary condition, the number of additional invariant polynomials strongly depends of the spectra of the one-qubit reduced density matrices, i.e., on the point in the polytope $\Psi(M)$. For $\alpha \in \Psi(M)$, the number of additional polynomials is given by the dimension of the reduced space $M_\alpha = \Psi^{-1}(\alpha)/K$. In [10] $\dim_{\mathbb{R}} \Psi^{-1}(\alpha)/K$ was analyzed for arbitrary $\alpha \in \Psi(M)$.

For multi-qubit systems the inequalities describing the polytope $\Psi(M)$ are known [22]. Denote by $\{p_i, 1 - p_i\}$ an increasingly ordered spectrum of the i -th reduced density matrix and by λ_i the shifted spectrum, $\lambda_i = \frac{1}{2} - p_i$. Then, $\Psi(M)$ is given by $0 \leq \lambda_i \leq \frac{1}{2}$ and $(\frac{1}{2} - \lambda_i) \leq \sum_{j \neq i} (\frac{1}{2} - \lambda_j)$. Methods used in [10] to compute the dimensions of spaces M_α , are different for points α belonging to the interior of $\Psi(M)$ and for points from the boundary of the polytope. For more than two qubits, the polytope is of full dimension, hence a generic K -orbit in the space of states M has the dimension of K [9]. Using the regularity of μ [23, 24] we get that for points α from the interior of the polytope the

dimension of the reduced space reads:

$$\begin{aligned}\dim M_\alpha &= \dim(\Psi^{-1}(\alpha)/K) = (\dim \mathbb{P}(\mathcal{H}) - \dim \Psi(\mathcal{H})) - \dim K = \\ &= ((2^{L+1} - 2) - L) - 3L = 2^{L+1} - 4L - 2.\end{aligned}\quad (12)$$

Points belonging to the boundary of $\Psi(M)$ can be grouped into three classes: (i) k of λ_l are equal to $\frac{1}{2}$, (ii) at least one of inequalities $(\frac{1}{2} - \lambda_l) \leq \sum_{j \neq l} (\frac{1}{2} - \lambda_j)$ is an equality, (iii) k of λ_l are equal to 0. In case (i), inequalities that yield $\Psi(M)$ reduce to an analogical set of inequalities for the $(L - k)$ -qubit polytope. Therefore, $\dim M_\alpha = ((2^{L-k+1} - 2) - (L - k)) - 3(L - k) = 2^{L-k+1} - 4(L - k) - 2$. States that are mapped to points that fall into case (ii) belong to the $K^\mathbb{C}$ -orbit through the L -qubit W -state, $[W] = |01 \dots 1\rangle + |101 \dots 1\rangle + \dots + |1 \dots 10\rangle$ [10]. As it is shown in [9], the closure of such an orbit is an almost homogeneous spherical space. Therefore, fibers of the momentum map are contained in K -orbits (see Section 3.1), i.e. $\dim M_\alpha = 0$. Case (iii), where k of λ_l are equal to 0, is the most difficult one, as it requires the use of some more advanced tools from the Geometric Invariant Theory (GIT) [8]. Here a key role is played by *stable states* [8, 25], i.e. states for which $\mu([\phi]) = 0$ and $\dim K \cdot [\phi] = \dim K$ [26]. For a symplectic action of a compact group \tilde{K} the existence of stable states implies that $\tilde{\mu}^{-1}(0)/\tilde{K} = \dim \mathbb{P}(\mathcal{H}) - 2\dim \tilde{K}$, where $\tilde{\mu}$ is the momentum map for the \tilde{K} -action. The strategy taken in [10] for case (iii) is the following. The group K can be divided into $K = K_1 \times K_2$, where $K_1 = SU(2)^{\times k}$, $K_2 = SU(2)^{\times (L-k)}$ and K_1 acts on the first k qubits. The action of K_1 yields the momentum map, which assigns to a state its first k one-qubit reduced density matrices. Therefore, $\mu_1^{-1}(0)$ consists of states whose first k reduced density matrices are maximally mixed, while the remaining $(L - k)$ reduced matrices are arbitrary. Further one constructs a state that is GIT stable with respect to the action of $K_1^\mathbb{C}$ on $\mathbb{P}(\mathcal{H})$. Hence, $\dim \mu_1^{-1}(0)/K_1 = \dim \mathbb{P}(\mathcal{H}) - 2\dim K_1 = 2^{L+1} - 6k - 2$. Furthermore, the quotient $\mu_1^{-1}(0)/K_1$ is a symplectic variety itself. Because the actions of K_1 and K_2 commute, we can consider the action of K_2 on $\mu_1^{-1}(0)/K_1$. The momentum map for K_2 acting on $\mu_1^{-1}(0)/K_1$ gives the remaining $L - k$ one-qubit reduced density matrices. The polytope of Ψ_2 is of full dimension, i.e. of dimension $L - k$. By the formula for the dimension of the reduced space for points from the interior of the polytope, we get:

$$\begin{aligned}((\dim \mu_1^{-1}(0)/K_1) - (L - k)) - \dim K_2 &= ((2^{L+1} - 6k - 2) - (L - k)) - 3(L - k) \\ &= 2^{L+1} - 4L - 2k - 2,\end{aligned}$$

which is the desired result for the case (iii).

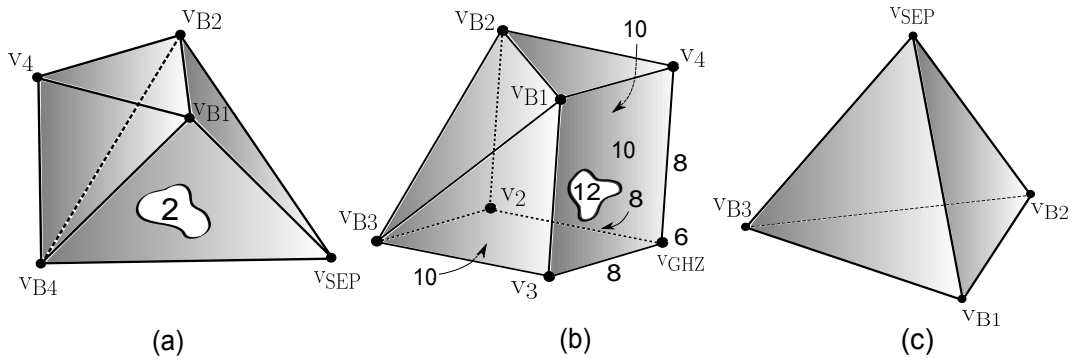


Figure 1: The three parts of the boundary of $\Psi(\mathbb{P}(\mathcal{H}))$ for four qubits. The numbers denote $\dim M_\alpha$. If the number is missing, then $\dim M_\alpha = 0$. Taken from [10].

4 SLOCC-equivalence of quantum states

A classification of states with respect to SLOCC [27] can be, as already mentioned, treated as a complementary one to the LU-equivalence characterization of quantum correlations. For reasons briefly explained below such a classification is not an easy task and its intricacies are still not fully understood. To a large extent the arising problems were fairly exhaustively explained in [28].

4.1 Invariant polynomials approach

For the considered multipartite systems reversible SLOCC operations correspond to elements of the complexification $G = K^{\mathbb{C}}$ of the local unitary group K , and two states are SLOCC equivalent if and only if they belong to the same G -orbit. Recall that the problem of K -equivalence is solvable by means of K -invariant polynomials. As the group G is reductive, (which means that it is a complexification of its maximal compact group - in this case the compact group K), the Hilbert-Nagata theorem [8] ensures that the ring of G -invariant polynomials is finitely generated, just like in the compact case of the local unitary group K . Nevertheless, the problem of G -equivalence turns out to be significantly different from the problem of K -equivalence. The essence of this difference is the fact that the group G is not compact and thus G -orbits do not have to be closed. For two vectors ϕ_1 and ϕ_2 on two non-intersecting orbits, the closures of the orbits can intersect. Since G -invariant polynomials are continuous functions, they are not able to distinguish between orbits $G.\phi_1$ and $G.\phi_2$ in such a case. It is only possible to distinguish between orbits whose closures have non-empty intersection, in particular between closed G -orbits. In purely topological language this ‘pathology’ can be linked to the fact that the orbit space M/G , i.e., the quotient space identifying states on the same orbit, is not longer a Hausdorff space - not every pair of points is separated by open sets. The G -equivalence of states is thus intimately linked to the structure of the orbit space resulting from the action of a non-compact reductive group on a vector space \mathcal{H} (equivalently on the projective space $\mathbb{P}(\mathcal{H})$).

4.2 Geometric Invariant Theory approach

Two orbits of $G.\phi$ and $G.\psi$ in \mathcal{H} are called c -equivalent if and only if there exists a sequence of orbits $G.\phi = G.v_1, G.v_2, \dots, G.v_n = G.\psi$ such that the closures of each two consecutive ones intersect, $\overline{G.v_k} \cap \overline{G.v_{k+1}} \neq \emptyset$. The relation of c -equivalence divides G -orbits into equivalence classes (c -classes). It turns out that every c -class contains exactly one closed G -orbit contained in the closure of every G -orbit belonging to the considered c -class. The equivalence classes are thus parametrized by closed G -orbits and G -invariant polynomials distinguish between G -orbits belonging to different c -classes [8]. Among all c -classes we distinguish those corresponding to the zero vector - they form the so-called null cone [29]. This class must be removed if we want to consider the quotient space at the projective level. After removing from the projective space $\mathbb{P}(\mathcal{H})$ points corresponding to vectors from the null cone we are left with so called *semistable* points, $\mathbb{P}(\mathcal{H})_{ss}$. Two points $x_1, x_2 \in \mathbb{P}(\mathcal{H})_{ss}$ are c -equivalent if there are vectors of $v_1, v_2 \in \mathcal{H}$, such that $x_1 = [v_1]$ and $x_2 = [v_2]$ and on the level of the Hilbert space $\overline{G.v_1} \cap \overline{G.v_2} \neq \emptyset$. The quotient space obtained from the semistable points by c -equivalence relation is denoted by $\mathbb{P}(\mathcal{H})_{ss} // G$ and is a projective algebraic variety. It is known in the literature under the name *GIT*

quotient [25] and we will call it *GIT space*. Points of the GIT space correspond to c -classes of semistable points and are in one-to-one correspondence with closed G -orbits. It turns out that every closed G -orbit in $\mathbb{P}(\mathcal{H})_{ss}$ contains exactly one K -orbit from $\mu^{-1}(0)$ [30]. Therefore we get the following equivalence $\mu^{-1}(0)/K \cong \mathbb{P}(\mathcal{H})_{ss} // G$, giving thus a description the GIT space in terms of the momentum map (see Figure 2)

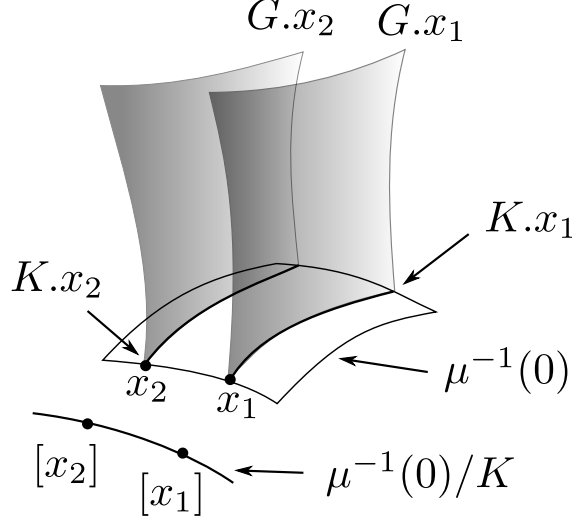


Figure 2: The idea of the GIT quotient construction, i.e. $\mu^{-1}(0)/K \cong \mathbb{P}(\mathcal{H})_{ss} // G$. Taken from [28].

The set of closed G -orbits is given by the action of G on $\mu^{-1}(0)$, i.e. $G.\mu^{-1}(0)$. Among the semistable points we distinguish the so-called stable points $\mathbb{P}(\mathcal{H})_s = \{x \in \mathbb{P}(\mathcal{H})_{ss} : \dim G.x = \dim G \text{ and } G.x \cap \mu^{-1}(0) \neq \emptyset\}$ [25]. The existence of a single stable point makes $\mathbb{P}(\mathcal{H})_s$ an open dense subset of $\mathbb{P}(\mathcal{H})_{ss}$, i.e. almost every semistable point is stable [31]. For the stable point $x \in \mathbb{P}(\mathcal{H})_s$ the c -equivalence class consists of exactly one closed G -orbit. For semistable but not stable points this class always consists of an infinite number of G -orbits.

Vectors belonging to the null cone, i.e. c -class whose closed G -orbit is the zero vector, may represent important states from the point of view of quantum correlations. For example, the W-state $|W\rangle = 1/\sqrt{3}(|110\rangle + |101\rangle + |011\rangle)$ and separable states belong to the null cone but their quantum properties are significantly different. Therefore we need a finer procedure dividing G -orbits, one that includes the GIT construction and also provides mathematically and physically well-defined stratification of the null cone. A key role is played here by the function $\|\mu\|^2 : \mathbb{P}(\mathcal{H}) \rightarrow \mathbb{R}$, i.e., the norm of the momentum map. It has a clear mathematical and physical interpretation. According to the definition given by Klyachko [32], the total variance of state $[v] \in \mathbb{P}(\mathcal{H})$ with respect to the symmetry group $K \subset SU(\mathcal{H})$ is given by

$$\text{Var}([v]) = \frac{1}{\langle v|v \rangle} \left(\sum_{i=1}^{\dim K} \langle v|\xi_i^2|v \rangle - \frac{1}{\langle v|v \rangle} \sum_{i=1}^{\dim K} \langle v|\xi_i|v \rangle^2 \right) = c - 4 \cdot \|\mu\|^2([v]), \quad (13)$$

where ξ_i form an orthonormal basis of algebra \mathfrak{k} and c is a $[v]$ -independent constant. The function $\|\mu\|^2([v])$ can be also expressed as the expectation value of the Casimir operator [33], $\mathcal{C}_2 = \sum_{i=1}^{\dim K} \xi_i^2$ for the irreducible representation of K on the symmetrized tensor product $\text{Sym}^2 \mathcal{H}$ [34]. In this case we have $\mathcal{C}_2^\vee = \sum_{i=1}^{\dim K} (\xi_i \otimes I + I \otimes \xi_i)^2$ and

$\frac{1}{\langle v|v \rangle^2} \langle v \otimes v | C_2^\vee | v \otimes v \rangle = 2c + 8 \|\mu\|^2([v])$. Finally, $\|\mu\|^2$ is directly related to the linear entropy, which is a linear function of the total variance.

A point $[v] \in \mathbb{P}([v])$ is a critical point of $\|\mu\|^2$ if it is a solution of $\mu([v]).v = \lambda v$ [28]. Critical points of $\|\mu\|^2$ can be therefore divided into two categories. The first includes all K -orbits belonging to $\mu^{-1}(0)$. These are called minimal critical points and for them $\|\mu\|^2$ reaches a global minimum. The minimal critical points correspond to states with maximum total variance and maximum linear entropy. The other critical points are given by some K -orbits in the null cone. For these points $\mu([v]) \neq 0$ and $\mu([v])v = \lambda v$. Therefore, in the null cone we distinguish G -orbits passing through the critical K -orbits.

The relationship between critical points of $\|\mu\|^2$, c -equivalence and GIT construction becomes clear if we consider the gradient flow of $-\|\mu\|^2$ [29]. The gradient of $-\|\mu\|^2$ is well defined as the projective space $\mathbb{P}(\mathcal{H})$ is a Kähler manifold, and therefore is equipped with a well defined metric determined by its Riemannian structure (see Appendix). The gradient flow is tangent to G -orbits and carries points towards critical K -orbits. Two points $x_1, x_2 \in \mathbb{P}(\mathcal{H})_{ss}$ are equivalent from the point of view of the gradient flow if they are taken by it to the same critical K -orbit. This definition is consistent with the c -equivalence definition. However, it is at the same time more general because it allows an extension of the concept of equivalence to the null cone. The situation in the null cone is more complex as the critical K -orbits do not need to be in one fiber of Ψ (recall that $\Psi : \mathbb{P}(\mathcal{H}) \rightarrow \mathfrak{t}$ is given by $\Psi(|\phi\rangle) = \mu(K.|\phi\rangle) \cap \mathfrak{t}_+$). Nevertheless, the polytope $\Psi(\mathbb{P}(\mathcal{H}))$ has a finite number of points $\{\alpha_i\}$ for which $\Psi^{-1}(\alpha_i)$ contains critical K -orbits. Let C_α denote the set of critical K orbits mapped by Ψ on $\alpha \in \mathfrak{t}_+$ and N_α be the set of all the points that are taken by gradient flow of $-\|\mu\|^2$ to C_α . The quotient space $N_\alpha // G$, obtained from N_α by dividing N_α by the equivalence relation induced from the gradient flow and the space C_α/K are isomorphic algebraic varieties (see Figure 3).

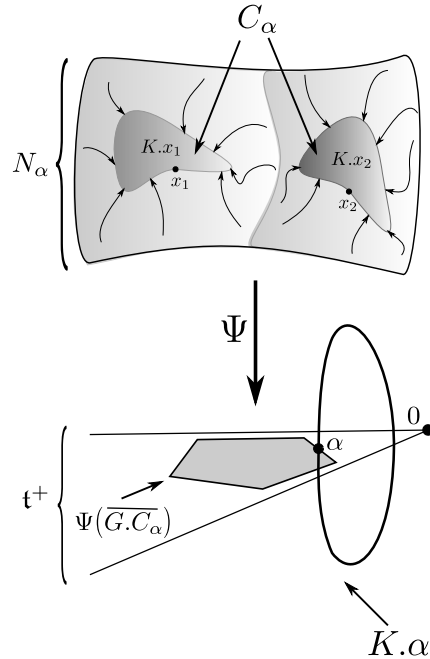


Figure 3: The sets N_α and C_α , with two exemplary critical K -orbits, $K.x_1$ and $K.x_2$. The arrows represent the gradient flow of $-\|\mu\|^2$. Taken from [28].

The above described construction is thus analogous to the GIT one. For $\alpha = 0$ we

get that $N_0 = \mathbb{P}(\mathcal{H})_{ss}$ and $C_0 = \mu^{-1}(0)$. Using the so defined equivalence relation we can think of a quotient space $\mathbb{P}(\mathcal{H})$ by G , i.e, the space identifying points on the same G -orbit, as of the space consisting of a finite number of projective algebraic varieties:

$$\mathbb{P}(\mathcal{H})/G \cong \bigcup_{\alpha} C_{\alpha}/K. \quad (14)$$

In the above formula we abused slightly the notation writing $\mathbb{P}(\mathcal{H})/G$, since, as explained above this is not a ‘good quotient’ from the point of view of the G -action.

The map Ψ has another important property, namely not only $\Psi(\mathbb{P}(\mathcal{H}))$ is a convex polytope but also the image of every G -orbit $\Psi(\overline{G.x})$ has this property [16]. A finite number of varieties C_{α}/K is the result of the fact that N_{α} can be equivalently defined as those $x \in \mathbb{P}(\mathcal{H})$ for which polytopes $\Psi(\overline{G.x})$ share the nearest point to the origin. But the momentum map convexity theorem for G -orbits ensures that the number of such polytopes is finite [35], so the number of manifolds C_{α} is also finite. Summarizing, we obtained the correspondence given in Table 1.

Table 1: A dictionary

G -orbit	SLOCC class of states
the momentum map μ	the map which assigns to a state $[v]$ the collection of its reduced one-particle density matrices
$ \mu ^2([v])$	the total variance of state $\text{Var}([v])$, linear entropy
closure equivalence class of orbits	family of asymptotically equivalent SLOCC classes
stable point	SLOCC family consists of exactly one SLOCC class
semistable but not stable point	SLOCC family consists of many SLOCC classes
$\Psi(\overline{G.[v]})$	SLOCC momentum polytope, collection of all possible spectra of reduced one-particle density matrices for $[u] \in \overline{G.[v]}$
strata N_{α}	group of families of SLOCC classes - all states for which SLOCC momentum polytopes have the same closest point to the origin
C_{α}	set of critical points of $\text{Var}([v])$ with the same spectra of reduced one-particle density matrices

In this way we achieved decomposition into a finite number of SLOCC classes determined by a single, easily accessible function $||\mu||^2$, which can be expressed in terms of the total variance of the state. In comparison with the approach using invariant polynomials the above presented one shows considerable advantages. It differentiates between states that clearly differ with respect to their correlation properties e.g., the W-state and separable states, which the invariant polynomial method puts to the same class. Moreover, invariant polynomials usually do not have clear physical meaning, in general they are not experimentally accessible, in contrast to the total variance, which can be measured. And finally, the number of different classes is finite, what enables an effective classification.

The space $\mathbb{P}(\mathcal{H})$ can be also divided into a finite number of generalized SLOCC classes using polytopes $\Psi(\overline{G.x})$, which in [36] are called entanglement polytopes. This is done by saying that two states are equivalent when their entanglement polytopes are the same.

Decomposition (14) is identical with that division up to the existence of polytopes that have a common closest point to the origin.

The key ingredient needed to obtain decomposition (14) is the knowledge of the critical K -orbits of $\|\mu\|^2$. In [28] they were found for two distinguishable and indistinguishable particles, three qubits and any number of two-state bosons. For four qubits it was shown that most classes found in [37] are c -equivalent with the class corresponding to $\mu^{-1}(0)$.

Critical $\alpha \in \Psi(\mathbb{P}(\mathcal{H}))$	State	$E(\phi)$
$\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$	Sep	0
$\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	TriSep	$\frac{1}{4}$
$\begin{pmatrix} -\frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}, \begin{pmatrix} -\frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}, \begin{pmatrix} -\frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$	$ W^{(3)}\rangle \otimes 1\rangle$	$\frac{1}{3}$
$\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	BiSep	$\frac{3}{8}$
$\begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$	W	$\frac{3}{8}$
$\begin{pmatrix} -\frac{1}{10} & 0 \\ 0 & \frac{1}{10} \end{pmatrix}, \begin{pmatrix} -\frac{1}{10} & 0 \\ 0 & \frac{1}{10} \end{pmatrix}, \begin{pmatrix} -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}, \begin{pmatrix} -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$	Φ_3	$\frac{9}{20}$
$\begin{pmatrix} -\frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}, \begin{pmatrix} -\frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}, \begin{pmatrix} -\frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	Φ_2	$\frac{11}{24}$
$\begin{pmatrix} -\frac{1}{14} & 0 \\ 0 & \frac{1}{14} \end{pmatrix}, \begin{pmatrix} -\frac{1}{14} & 0 \\ 0 & \frac{1}{14} \end{pmatrix}, \begin{pmatrix} -\frac{1}{14} & 0 \\ 0 & \frac{1}{14} \end{pmatrix}, \begin{pmatrix} -\frac{1}{7} & 0 \\ 0 & \frac{1}{7} \end{pmatrix}$	Φ_1	$\frac{27}{56}$
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	GHZ	$\frac{1}{2}$

Table 2: One-qubit reduced density matrices for critical states of four qubits. The listed states are: $|\text{TriSep}\rangle = \frac{1}{\sqrt{2}}|11\rangle \otimes (|00\rangle + |11\rangle)$, $|\text{BiSep}\rangle = \frac{1}{\sqrt{2}}|1\rangle \otimes (|000\rangle + |111\rangle)$, $|W\rangle = \frac{1}{2}(|1110\rangle + |1101\rangle + |1011\rangle + |0111\rangle)$, $|\Phi_3\rangle = \sqrt{\frac{3}{10}}(|1101\rangle + |1110\rangle) + \sqrt{\frac{2}{5}}|0011\rangle$, $|\Phi_2\rangle = \frac{1}{2\sqrt{3}}(|1011\rangle + |1110\rangle) - \frac{1}{2}(|0101\rangle + |0011\rangle) + \frac{1}{\sqrt{3}}|0110\rangle$, $|\Phi_1\rangle = \sqrt{\frac{3}{14}}(|0011\rangle + |0101\rangle + |1001\rangle) + \sqrt{\frac{5}{14}}|1110\rangle$, $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$.

4.3 Critical points of $\|\mu\|^2$ for many qubits

The critical points of $\|\mu\|^2$, or of the linear entropy, play a key role in understanding the generalized SLOCC classes. Finding the critical states by direct application of the definition is a computationally difficult task, as it requires solving the equation $\mu([v]).v = \lambda v$, i.e. finding eigenvectors and eigenvalues of a matrix depending nonlinearly on a vector it acts on, rendering thus seemingly straightforward eigenvectors-eigenvalues problem effectively nonlinear. In [38] a slightly more tractable method on an interplay between momentum maps for abelian and non-abelian Lie groups was proposed.

For a compact group K , we denote by T its maximal torus, which is a maximal compact and connected abelian subgroup. For example, when $K = SU(N)$, a maximal torus consists of unitary diagonal matrices with the determinant one. A momentum map $\mu_T : M \rightarrow \mathfrak{t}$ for the action of T on M is given by the composition of $\mu : M \rightarrow \mathfrak{k}$ with the

projection on the Cartan subalgebra $\mathfrak{t} = \text{Lie}(T)$. Therefore, we have $\mu([v]) = \mu_T([v]) + \alpha$, where $\alpha \in \mathfrak{t}^\perp$. By the convexity theorem, $\mu_T(M)$ is a convex polytope. For abelian groups, the convexity theorem specifies the vertices of the polytope [19]. The vertices are given by the set of weights $\mathbb{A} = \mu_T(M_T)$, where M_T are the fixed points for the action of T on M^3 . The critical points of $\|\mu_T\|^2$, must satisfy a similar condition as the critical points of $\|\mu\|^2$, i.e. $\mu_T([v]).v = \lambda v$. Therefore, for $\beta \in \mu_T(M)$ a point $[v]$ is a critical point iff $\beta.v = \lambda v$, i.e. $[v]$ is a fixed point for $T_\beta = \{e^{t\beta} : t \in \mathbb{R}\}$ and $\mu_T([v]) = \beta$. The fixed-point set M_{T_β} for the action of T_β is not a symplectic variety. However, weakening the definition of M_{T_β} by demanding points $[v]$ to satisfy $\langle \mu_T([v]), \beta \rangle = \langle \beta, \beta \rangle$ instead of being mapped to β , we get a symplectic variety Z_β . Points from Z_β are sent by μ_T to the hyperplane that is perpendicular to β and contains β [26]. Clearly, $M_{T_\beta} \subset Z_\beta$. The set Z_β is a T -invariant symplectic variety, hence, by the convexity theorem, we have that $\mu_T(Z_\beta)$ is a convex polytope, which is spanned by a subset of weights from \mathbb{A} . The definition of Z_β implies that β is the closest to zero point of this polytope. In other words, $[v] \in M$ is a critical point of $\|\mu_T\|^2$ iff it is mapped to a minimal convex combination of weights, β , and $[v] \in Z_\beta$ [26].

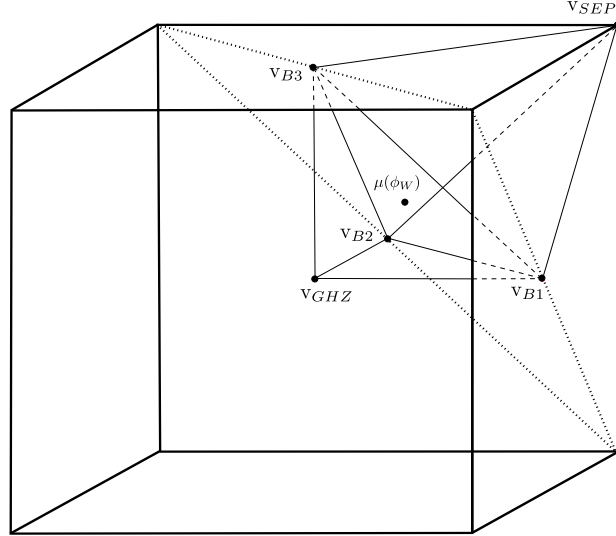


Figure 4: Minimal weight combinations for three qubits. Point v_{GHZ} is the image of state $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, points v_{Bi} correspond to the biseparable states and $|\phi_W\rangle = \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle)$. Taken from [38].

Function $\|\mu\|^2$ is K -invariant, therefore we can restrict our consideration to critical points satisfying $\mu([v]) \in \mathfrak{t}_+$. For such states we have $\mu([v]) = \mu_T([v])$ and $[v]$ is a critical point of $\|\mu\|^2$ iff it is a critical point of $\|\mu_T\|^2$. Let us denote by \mathcal{B} the set of all minimal combinations of weights from \mathbb{A} that belong to \mathfrak{t}_+ . Then, a state $[v]$ is a critical one if and only if $\mu([v]) \in \mathcal{B}$ and $[v] \in Z_\beta$. Critical sets are therefore of the form $C_\beta = K.(Z_\beta \cap \mu^{-1}(\beta))$, where $\beta \in \mathcal{B}$. In [38] the above reasoning was applied to compute the critical points of the linear entropy for pure states of L qubits. Set \mathbb{A} is the image under μ of the basis states $B = |i_1, \dots, i_L\rangle$, where $i_k \in \{0, 1\}$, hence $\#\mathbb{A} = 2^L$. We discuss the algorithm of finding the minimal combinations of weights and list the results up to

³A point $x \in M$ is fixed by the action of T iff $\forall t \in T. t.x = x$

$L = 5$ (the construction of the set of minimal combinations of weights is shown on figure 4). We also show that for $\beta \in \mathcal{B}$, the set $Z_\beta = \mathbb{P}(\mathcal{S})$, where \mathcal{S} is spanned by the basis states whose weights span β . Moreover, we show when sets C_β are nonempty and for each $\beta \in \mathcal{B}$ we describe a construction of a state that is mapped to β . We conclude that the number of critical values of the linear entropy grows super-exponentially with L .

5 Geometric and topological characterization of CQ and CC states

In previous sections we discussed applications of the momentum map in two significant problems of the theory of quantum correlations, LU and SLOCC equivalence of pure states. In [39] it is shown how methods of symplectic and algebraic geometry can be applied to some concrete problems involving mixed states, namely to exhibit geometric and topological aspects of quantum correlations for separable (non-entangled) states. The existence of quantum correlations for multipartite separable mixed states can be regarded as one of the most interesting quantum information discoveries of the last decade. In 2001 Ollivier and Żurek [2] and independently Henderson and Vedral [3] introduced the notion of quantum discord as a measure of the quantumness of correlations. Quantum discord is always non-negative [40]. The states with vanishing quantum discord are called *pointer states*. They form the boundary between classical and quantum correlations [40]. Bipartite pointer states can be identified with the so-called classical-quantum, CQ states [40]. An important subclass of CQ states are classical-classical, CC states [41].

For $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A = \mathbb{C}^{N_1}$ and $\mathcal{H}_B = \mathbb{C}^{N_2}$ a state is CC if it can be written as $\rho = \sum_{i,j} p_{ij} |i\rangle\langle i| \otimes |j\rangle\langle j|$, where $\{|i\rangle\}_{i=1}^{N_1}$ is an orthonormal basis in \mathcal{H}_A and $\{|j\rangle\}_{j=1}^{N_2}$ in \mathcal{H}_B . A state ρ is a CQ state if it can be written as $\rho = \sum_i p_i |i\rangle\langle i| \otimes \rho_i$, where $\{\rho_i\}_{i=1}^{N_1}$ are the density matrices on \mathcal{H}_B . Both classes are of measure zero in \mathcal{H} [42]. For pure states, separable states are exactly zero-discord states. It was shown in [5] that pure separable states are geometrically distinguished in the state space and belong to the unique symplectic K -orbit in $\mathbb{P}(\mathcal{H})$. For mixed states, already for two particles it is easy to see that there are infinitely many symplectic K -orbits and there are separable states through which K -orbits are not symplectic. Thus a simple extension of the results of [5], even for two-particle mixed states is not possible. In [39] four facts concerning geometric and topological characterizations of CC and CQ states are shown. They extend results of [5] to mixed states: (1) the set of CQ states is the closure of all symplectic orbits of $K = SU(N_1) \times I_{N_2}$ action, (2) the set of CC states is the closure of all symplectic orbits of $K = SU(N_1) \times SU(N_2)$ action, (3) the set of CQ states is exactly the set of $K = SU(N_1) \times I_{N_2}$ orbits whose Euler-Poincaré characteristics χ do not vanish, (4) the set of CC states is exactly the set of $K = SU(N_1) \times SU(N_2)$ orbits whose Euler-Poincaré characteristics χ do not vanish.

The space of all density matrices is not a symplectic space (the symplectic form is degenerate). Nevertheless, as already mentioned in the introduction (see also Appendix), the set of density matrices with the fixed spectrum \mathcal{O}_ρ , which is the adjoint orbit of $SU(\mathcal{H})$ through ρ is symplectic. Therefore, the action of the above given groups K on \mathcal{O}_ρ leads to existence of the momentum map $\mu : \mathcal{O}_\rho \rightarrow \mathfrak{k}$ [7]. In order to check if a given orbit $K \cdot \sigma$ (the action of K on $\sigma \in \mathcal{O}_\rho$ is the adjoint action) is or is not symplectic it is enough to consider the restriction of the momentum map μ to $K \cdot \sigma$. Then $K \cdot \sigma$ is symplectic if this restriction is bijective. The computational conditions for μ to be bijective are given in the

Kostant-Sternberg theorem [43]. Let us note that since $K.\rho$ is mapped by μ onto adjoint orbit in \mathfrak{k} , non-symplecticity of $K.\rho$ (the degeneration of symplectic form on $K.\rho$) can be measured as $D(K.\rho) = \dim K.\rho - \dim \text{Ad}_K \mu(\rho)$. For two qubits the CC states, in a fixed basis, form a 3-dimensional simplex and therefore it is possible to see how the closure of the symplectic $K = SU(N_1) \times SUN_2$ orbits forms the set of CC states (figure. 5, 6 and 7). In [39] we also discuss existence of Kähler structure and show that it is present on all considered symplectic K -orbits.

For finding Euler-Poincaré characteristics χ we use the Hopf-Samelson theorem [44]. This theorem says that for action of a compact group K on a manifold M the Euler-Poincaré characteristics χ of the orbit K/K_x passing through $x \in M$ is given by:

1. If the maximal torus T of K is contained in K_x then $\chi(K/K_x) = \frac{|W_K|}{|W_{K_x}|}$, where W_K and W_{K_x} are Weyl groups of K and K_x respectively.
2. Otherwise, $\chi(K/K_x) = 0$.

In [39] it is shown that orbits of the discussed groups through CC and CQ states are the only orbits with stabilizer subgroups containing maximal torus. The ranks of the Weyl groups are calculated and a formula for χ is given.

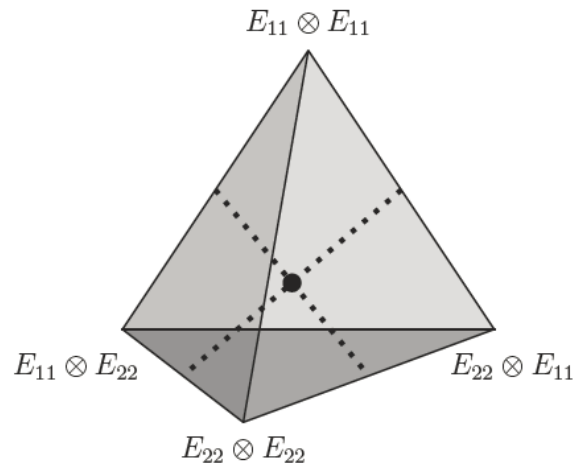


Figure 5: Dimensions of orbits through CC states of two qubits. The large dot: $\dim K.\rho = 0$, the dotted lines: $\dim K.\rho = 2$, elsewhere: $\dim K.\rho = 4$. Taken from [39].

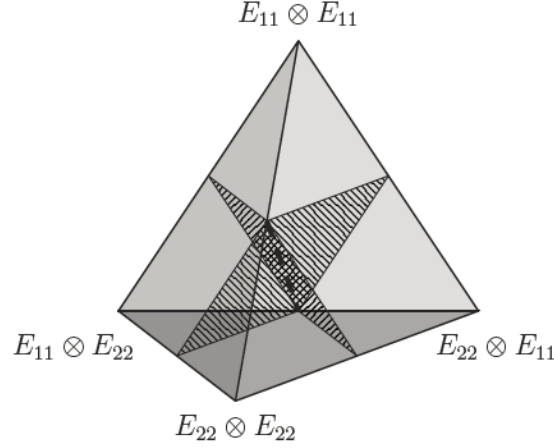


Figure 6: Ranks of $\omega|_{K,\rho}$ for orbits through CC states of two qbits. The thick dashed line: $\text{rk } \omega|_{K,\rho} = 0$, the lined surfaces: $\text{rk } \omega|_{K,\rho} = 2$, elsewhere: $\text{rk } \omega|_{K,\rho} = 4$. Taken from [39].

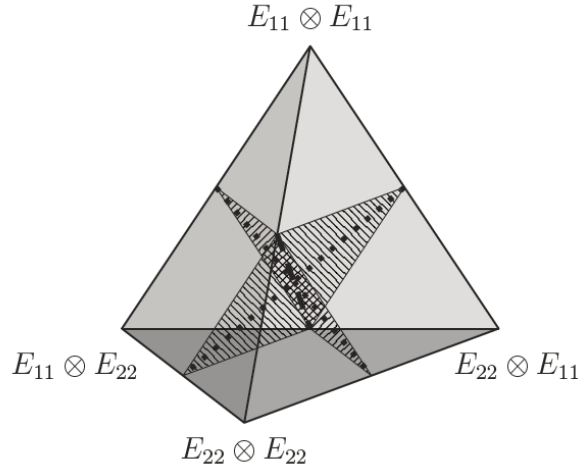


Figure 7: Degrees of degeneracy of $\omega|_{K,\rho}$ for orbits through CC states of two qbits. The thick dashed line: $D(K,\rho) = 4$, the lined surfaces: $D(K,\rho) = 2$, the dotted lines and elsewhere: $D(K,\rho) = 0$. Taken from [39].

6 Summary and outlook

One of the basic problems in the theory of quantum correlations is the classification of states with respect to local operations performed independently on subsystems of a given system. Two such classes are particularly interesting, (1) local unitary (LU) operations, and (2) SLOCC - Stochastic Local Operations with Classical Communication. Mathematically, these operations correspond to the action of some compact group $K \subset SU(\mathcal{H})$ in case (1) and its complexification $G = K^{\mathbb{C}}$ in case (2). The space of pure states (after neglecting the global phase) is the projective space $\mathbb{P}(\mathcal{H})$, and for mixed states, the space of isospectral density matrices is an adjoint orbit of the unitary group $SU(\mathcal{H})$ action. In both cases, these spaces have a natural geometric (Kähler) structure, and therefore in

particular they are symplectic manifolds. Since the action of compact group on M preserves the symplectic structure there exists the momentum map. In the cases considered here, it assigns to a state of L particles its reduced one-particle density matrices, and therefore is directly related to partial trace over $L - 1$ particles. This observation opened new possibilities for the analysis employing tools and methods of symplectic and algebraic geometry.

We showed how such an approach can be applied to analyze/solve the following problems.

- When is information contained in one-particle reduced density matrices sufficient to solve the problem of LU-equivalence?
- What are obstacles in the case when such an information is insufficient, i.e., when the LU-equivalence can not be established upon examining spectra of reduced matrices?
- How many additional invariant polynomials (except those directly derived from one-qubit density matrices) are needed to solve the LU-equivalence problem? How his number depends on the spectra of reduced matrices?
- How to classify states under SLOCC operations? Such a classification should be effective, use straightforwardly calculated quantities and yield a finite number of generalized SLOCC classes.
- How to characterize geometrically mixed states with zero discord, more specifically of CC and CQ states?

The proposed methods are, in principle applicable, to distinguishable as well as indistinguishable particles, but concrete results for the latter are scarce what shows a direction for further studies.

Another direction of future research may concentrate on looking for other ways to identify critical orbits important for SLOCC equivalence, for example by employing other approach to stratification described in Section 4.2 [45]. It should allow moving a state to the appropriate critical orbit case a one parameter SLOCC subgroup and thus have a clear operational meaning since one could decide to which class a state belongs by acting on it by a one parameter family of SLOCC (hence physical) operations.

Appendix. Group actions on symplectic manifolds. Momentum maps

A symplectic manifold is an abstract generalization of a phase-space in classical mechanics. In the following we will invoke notions known in classical mechanics to illustrate some features stemming from this generalizations. Mathematically, a symplectic manifold is a smooth manifold M equipped with a closed, $d\omega = 0$, and non-degenerate differential two-form ω . The nondegeneracy means that if X is a tangent field to M such that at each $p \in M$ and each tangent vector Y at p we have $\omega(X, Y) = 0$ then X must vanish everywhere on M . It implies that a symplectic manifold is always even-dimensional.

A *Kähler manifold* is a complex symplectic manifold equipped with additional Riemannian structure g (a metric, i.e. positive-definite symmetric two-form), such that all three structures (complex, symplectic and Riemannian) are compatible *vis.*, $g(u, v) = \omega(u, iv)$,

or, equivalently, $\omega(u, v) = g(iu, v)$. It also means that $h := g + i\omega$ is a Hermitian metric i.e., $h(iu, iv) = h(u, v)$

A symplectic manifold (M, ω) is a natural geometric structure for classical Hamiltonian mechanics. A symplectic manifold is a classical mechanical phase-space equipped with the structure of Poisson brackets. They are defined in terms of ω in the following way. For an (appropriately smooth) function F on the phase space we define a tangent vector field X_F *via*

$$dF = \omega(X_F, \cdot), \quad (15)$$

i.e. action of the two form ω on the vector field X_F gives a one form (as it should) equal to the differential of F . Then we set for the Poisson bracket of two functions F and H ,

$$\{F, H\} = \omega(X_F, X_H) \quad (16)$$

The dynamics of a system is determined by a Hamilton function H *via* canonical (Hamilton) equations of motion

$$\frac{d}{dt}F = \{F, H\} \quad (17)$$

for an arbitrary phase-space function F . Using now the definitions of the fields X_H , X_F and the Poisson brackets we have.

$$\frac{d}{dt}F = \{F, H\} = \omega(X_F, X_H) = dF(X_H) = X_H(F),$$

hence the vector field X_H determines at each point $p \in M$ the direction in which this point moves under the dynamics generated by the Hamilton function H .

The momentum map appears always when a Lie group K acts on a symplectic manifold preserving the symplectic structure. In classical mechanics this corresponds to a situation when we have a group of canonical (i.e. preserving the Poisson brackets) symmetries. In the paper K is always a connected compact semi-simple matrix Lie group, in fact a subgroup of a unitary group. Let us denote the action of K on M by $x \mapsto \Phi_g(x)$, $x \in M$, $g \in K$. For each element of the Lie algebra $\xi \in \mathfrak{k} = \text{Lie}(K)$ of the group K we define the *fundamental vector field* $\hat{\xi} \in \chi(M)$

$$\hat{\xi}(x) = \left. \frac{d}{dt} \right|_{t=0} \Phi_{\exp t\xi}(x), \quad (18)$$

where $\chi(M)$ - smooth vector fields on M . Invoking again the classical mechanical origin of the presented concepts we may call ξ a generator of the one dimensional subgroup of symmetries. The vector field $\hat{\xi}$ at $p \in M$ points to the direction in which the phase-space point p moves under the action of this one-parameter subgroup.

Under additional conditions⁴ fulfilled in all cases considered here, there is a well defined function μ_ξ , such that $d\mu_\xi = \omega(\hat{\xi}, \cdot)$. Upon referring to the definitions above, we see thus that μ_ξ plays a role of a Hamilton function for the ‘motion’ of phase-space points under the one-parameter subgroup in question, where the parameter plays a role of the time. In classical mechanics $\mu_x i$ would be called a generating function for the one parameter group.

The functions μ_ξ can be chosen to be linear in $\xi \in \mathfrak{k}$ and thus define the unique map $\mu : M \rightarrow \mathfrak{k}^*$, where \mathfrak{k}^* is the dual vector space of \mathfrak{k} , i.e. the space of linear forms on the

⁴ K is semi-simple and M has the trivial first de Rham cohomology group

linear space \mathfrak{k} . Hence, $\mu(x)$ is defined by $\langle \mu(x), \xi \rangle = \mu_\xi(x)$, where $\langle \alpha, \zeta \rangle$ denotes the action of a form α on a vector ζ . The function $\mu : M \rightarrow \mathfrak{k}^*$ is called the *momentum map*.

The group K acts also on its Lie algebra \mathfrak{k} by the adjoint action $\text{Ad}_g \xi = g \xi g^{-1}$. It has its natural dual on the dual space \mathfrak{k}^* , the coadjoint action

$$\langle \text{Ad}_g^* \alpha, \xi \rangle = \langle \alpha, \text{Ad}_{g^{-1}} \xi \rangle = \langle \alpha, g^{-1} \xi g \rangle, \quad g \in K, \quad \xi \in \mathfrak{k}, \quad \alpha \in \mathfrak{k}^*. \quad (19)$$

The set of points obtained by the action of a group on a manifold on a particular point p we call the orbit of the group (through the point p). In particular the set $\mathcal{O}_p = \{q = \Phi_g(p) : g \in K\} \subset M$ is the orbit through a point p of the action of K on M , and $\Omega = \{\beta = \text{Ad}_g^* \alpha\} \subset \mathfrak{k}^*$ - a coadjoint orbit of K through $\alpha \in \mathfrak{k}^*$. To shorten the notation we use occasionally $K.x$ to denote the orbit of K through x , whatever the action of K and the manifold on which it acts are.

Each coadjoint orbit is equipped with a natural symplectic structure given by the so-called Kirillov-Kostant-Souriau form. To define it let us first construct the fundamental vector field for the coadjoint action of the one-parameter subgroup of K generated by a Lie-algebra element ξ at some point $\alpha \in \mathfrak{k}$,

$$\tilde{\xi}(\alpha) = \left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{\exp(t\xi)}^* \alpha. \quad (20)$$

Now we define the Kirillov-Kostant-Souriau form at each point α on a coadjoint orbit by

$$\omega(\tilde{\xi}(\alpha), \tilde{\zeta}(\alpha)) = \langle \alpha, [\xi, \zeta] \rangle, \quad (21)$$

where $[\cdot, \cdot]$ is the Lie bracket in the Lie algebra \mathfrak{k} . For a semi-simple K the momentum map is equivariant, i.e. $\mu(\Phi_g(x)) = \text{Ad}_g^* \mu(x)$ for any $x \in M$ and $g \in K$. Orbits of K -action on M are therefore mapped by μ onto orbits of the coadjoint action in \mathfrak{k}^* .

Coadjoint and adjoint orbits (i.e. the orbits of the adjoint action of K on its Lie algebra \mathfrak{k}) can be identified in all cases we consider in the paper (i.e. for K compact as a subgroup of a unitary group) using a non-degenerate scalar product on \mathfrak{k} defined by $(\xi, \zeta) = -\text{Tr} \xi \zeta$, i.e, identifying a linear form $\alpha \in \mathfrak{k}^*$ with the vector $\xi \in \mathfrak{k}$ such that $(\xi, \zeta) = \langle \alpha, \zeta \rangle$ for all $\zeta \in \mathfrak{k}$. We use this identification throughout the paper and therefore we treat the moment map μ as a map from M to \mathfrak{k} rather than \mathfrak{k}^* .

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